Bonus malus systems and finite and continuous time ruin probabilities in motor insurance

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Goal

- Evaluate (numerically) the ruin probability in continuous and finite time for a Compound Poisson risk process,
- The premium is received at a constant rate during each year,
  - P_0 - Constant for $0 \leq t \leq n$,
  - Changing every year - Bonus Malus System (BMS).
- Model for large portfolios,
- Use calculation and simulation,
  - Simulation of the annual aggregate claims amounts,
  - Estimation of the premiums amounts received each year,
  - Calculation of the end of period (year) surplus,
  - Calculation of the ruin probability during the period.

The surplus process

$U(t)$ is the insurer's surplus at time t, $0 \leq t \leq n$

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t)\ , \ \ i = 1, \ldots, n$$

i is such that $t \in [i-1, i)$ and $\sum_{j=1}^{0} P_j = 0$

- u is the initial surplus
- P_i is the premium charged in year i (function of the past number of claims)
- $S(t)$ is the aggregate claims up to time t

Y_i is the aggregate claims in year i, $Y_i = S(i) - S(i-1)$.
$\{Y_i\}_{i=1}^{n}$ is a sequence of i.i.d., random variables, with a **compound Poisson distribution whose first three moments exist**.

Probability of ruin within the year

$\psi(u(i-1),1,u(i))$ is the probability of ruin within the year i, given the surplus $u(i-1)$ at the start of the year and the surplus $u(i)=y$ at the end of the year and the rate of premium income p_i during the year.
Following Dickson and Waters (2006).

$\Delta(u(i-1),1,z)$ is the probability, that starting from the surplus $u(i-1)$, ruin will not occur during the year and the surplus at the end of the year is greater than y.

$$\Delta(u(i-1),1,y) = \int_y^\infty (1-\psi(u(i-1),1,z)\, f(u(i-1)+p_i-z,1)\, dz$$

then:

$$\psi(u(i-1),1,y) = 1 + \frac{1}{f(u(i-1)+p_i-y,1)} \frac{d}{dy}\Delta(u(i-1),1,y)$$

Method: Dickson and Waters (2006)

$$\psi(u(i-1),1,u(i)) =$$

$$\frac{\int_{s=0}^{1-u(i)/p_i} f(u(i-1)+p_i s, s)\frac{u(i)}{(1-s)} f(p_i(1-s)-u(i),1-s)ds}{f(u(i-1)+p_i-u(i),1)}$$

$$+ \quad \frac{f(u(i-1)+p_i-u(i),1-u(i)/p_i)\exp(-\lambda u(i)/p_i)}{f(u(i-1)+p_i-u(i),1)}$$

- Exact formula to calculate $\psi(u(i-1),1,u(i))$;
- We need values for $f(\cdot,s)$ to s from 0 to 1;
- Time consuming task for high λ.

$$\Downarrow$$

We need a numerical method to approximate $\psi(u(i-1),1,u(i))$.

Method: Dickson and Waters (2006)

- $f(\cdot, s)$ is the probability density function of $S(s)$ for $0 < s \leq 1$. We are going to approximate it by a translated gamma distribution chosen to match the first three moments.
- α, β and κ are such that:

$$\begin{aligned}
\frac{2}{\sqrt{\alpha}} &= \frac{E[(Y_i - E[Y_i])^3]}{Var[Y_i]^{3/2}} \\
\frac{\alpha}{\beta^2} &= Var[Y_i] \\
\frac{\alpha}{\beta} + \kappa &= E[Y_i]
\end{aligned}$$

- $H(s) \sim G(\cdot; \alpha s, \beta)$
- For $0 < s \leq 1$ the random variable $H(s) + \kappa s$ has a translated gamma distribution and the first three moments are the same as $S(s)$.

In practice:

$$\begin{array}{rcl} f(x,s) & \text{replaced by} & f_G(x-\kappa s, s) \\ \exp(-\lambda t) & \text{replaced by} & F_G(-\kappa t, t) \end{array}$$

$$\psi(u(i-1),1,u(i)) \sim \psi_{TG}(u(i-1),1,u(i))$$

$$\psi_{TG}(u(i-1),1,u(i)) =$$

$$\frac{\int_{s=0}^{1-u(i)/p_i} f_G(u(i-1)+(p_i-\kappa)s,s)\frac{u(i)}{(1-s)}f_G((p_i-\kappa)(1-s)-u(i),1-s)ds}{f_G(u(i-1)+p_i-\kappa-u(i),1)}$$

$$+\frac{f_G(u(i-1)+(p_i-\kappa)(1-\frac{u(i)}{p_i}),1-\frac{u(i)}{p_i})F_G(-\kappa u(i)/p_i,u(i)/p_i)}{f_G(u(i-1)+p_i-\kappa-u(i),1)}$$

In practice:

- Approximation based on three moments;
- There are fast algorithms for gamma distributions;
- Fast results.

Bonus Malus Systems

- BMS are a posterior rating systems correcting the premiums according to past claim experience,
- Reduce the premium for insureds with low claim frequency rate and raise it for insureds with high claim frequencies,
- BMS can be modeled using Markov Chains,
- BMS are characterized by set of Transiton Rules, Initial Class and Premium Scale.

What is the impact of BMS or/and Premium Scale in the Probability of Ruin?

Assumptions

For a better perception of the following results, let us state the assumptions under which this work is based:

- The portfolio is **homogeneous** with respect to claim severities;
- The portfolio is **heterogeneous** with respect to claim frequencies, following a mixed Poisson distribution;
- We consider an **homogeneous claim frequency in each bonus malus level**, following a Poisson distribution with parameter λ_j;
- The portfolio is **closed** for ingoing and outgoing of policyholders.

Procedure

- starting at $u = U(0)$;
- simulate the aggregate claims, $\{y_i\}_{i=1}^{n}$,

  (using the translated gamma approximation, n. of claims in class j is Poisson distributed with parameter λ_j)
- calculate p_i (p_1 known), (Expected value principle, Commercial Scale, 6 Optimal Scales)
- calculate $u(i) = u(i-1) + p_i - y_i$;
- calculate $\psi_{TG}(u(i-1), 1, u(i))$;
  - If $u(i) < 0$ for any i, $i = 1, \ldots, n$ then ruin has occurred $\rightarrow \psi_j(u, n) = 1$ and we start another run $j + 1$.
  - If $u(i) \geq 0$ for all i, $i = 1, \ldots, n$ calculate $\psi(u(i-1), 1, u(i))$...
- $$\psi_j(u, n) = 1 - \prod_{i=1}^{n} \Big(1 - \psi_{TG}(u(i-1), 1, u(i))\Big).$$

Mean of the N runs $\rightarrow$ final estimate of $\psi(u, n)$.

Sample standard deviation $\rightarrow$ approximate confidence intervals for the estimate of $\psi(u, n)$.

Example - Portuguese Insurer

Claims amount	Y	$E(Y)$	1,766
		$V(Y)$	71,097,953
		$\gamma(Y)$	35.1
		$\gamma_2(Y)$	1,816

No. of claims $N \sim Poisson(\Lambda)$,
$\Lambda \sim InverseGaussian(\mu = 0.082401, \eta = 0.130271)$,
Total policies 442,490,

Time horizon $n = 10$,
No. runs 50,000

Example

$T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	p_0	-	-	p_1	-	-	p_2	-	-	p_3	-	-	p_4	-	-	p_5	-	$1-\sum_{i=0}^{5} p_i$
2	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	-	p_4	-	-	p_5	$1-\sum_{i=0}^{5} p_i$
3	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	-	p_4	-	-	$1-\sum_{i=0}^{4} p_i$
4	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	-	p_4	-	$1-\sum_{i=0}^{4} p_i$
5	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	-	p_4	$1-\sum_{i=0}^{4} p_i$
6	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	-	$1-\sum_{i=0}^{3} p_i$
7	-	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	-	$1-\sum_{i=0}^{3} p_i$
8	-	-	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	p_3	$1-\sum_{i=0}^{3} p_i$
9	-	-	-	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	-	$1-\sum_{i=0}^{2} p_i$
10	-	-	-	-	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	-	$1-\sum_{i=0}^{2} p_i$
11	-	-	-	-	-	-	-	-	-	p_0	-	-	-	p_1	-	-	p_2	$1-\sum_{i=0}^{2} p_i$
12	-	-	-	-	-	-	-	-	-	-	p_0	-	-	-	p_1	-	-	$1-\sum_{i=0}^{1} p_i$
13	-	-	-	-	-	-	-	-	-	-	-	p_0	-	-	-	p_1	-	$1-\sum_{i=0}^{1} p_i$
14	-	-	-	-	-	-	-	-	-	-	-	-	p_0	-	-	-	p_1	$1-\sum_{i=0}^{1} p_i$
15	-	-	-	-	-	-	-	-	-	-	-	-	-	p_0	-	-	-	$1-p_0$
16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	p_0	-	-	$1-p_0$
17	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	p_0	-	$1-p_0$
18	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	p_0	$1-p_0$

Example

Premiums	$P_0 = P = (1+\theta)E[Y]E[N_i], i \in \{1, \ldots, n\}$,
	C$=P\times$ Commercial Scale,
	N$=P\times$ Norberg Scale,
	NL$=P\times$ Gilde and Sund (1989) applied to N,
	NG$=P\times$ Andrade e Silva and Centeno (2005) applied to N,
	B$=P\times$ BorganScale,
	BL$=P\times$ Gilde and Sund (1989) applied to B,
	BG$=P\times$ Andrade e Silva and Centeno (2005) applied to B,
Initial Surplus	varying with the Premium type
Safety loading	$\theta = 0.8$

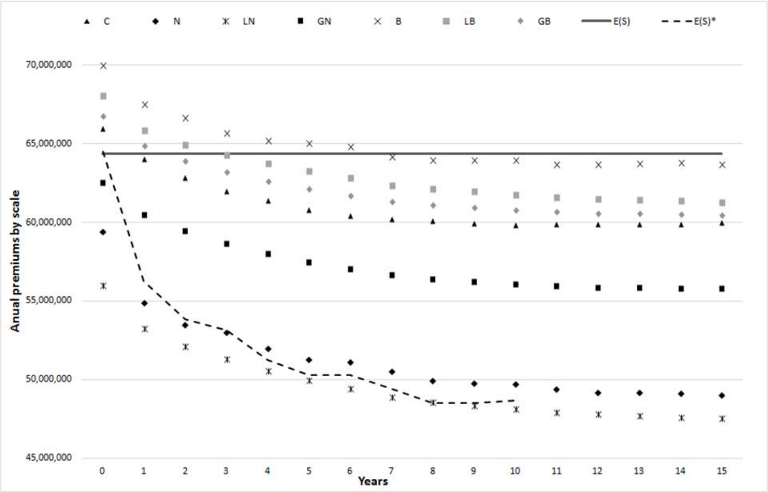
Example - Portuguese data

j	C	N	NL	NG	B	LB	GB	N. Policies	$\hat{\lambda}_j$
1	45	33.4	32.4	41.0	48.8	46.0	45.7	174,173	0.034516
2	45	48.0	39.9	45.3	58.0	52.0	49.8	109,113	0.072883
3	50	49.5	47.4	50.0	60.1	58.0	54.4	42,736	0.076425
4	55	51.1	54.9	55.2	62.3	64.0	59.3	29,134	0.080265
5	60	66.5	62.4	61.0	63.7	70.0	64.7	23,730	0.126855
6	65	69.9	69.9	67.3	66.2	76.0	70.6	4,241	0.135954
7	70	74.6	77.5	74.3	68.1	82.0	77.0	2,759	0.148393
8	80	87.3	85.0	82.0	69.9	88.0	84.0	24,829	0.181802
9	90	92.9	92.5	90.6	72.3	94.0	91.7	11,747	0.195919
10	100	100.0	100.0	100.0	100.0	100.0	100.0	166	0.213730
11	110	109.8	107.5	110.4	105.6	106.0	109.1	2,882	0.237433
12	120	117.0	115.0	121.9	113.0	112.0	119.0	7,632	0.255984
13	130	125.3	122.5	134.6	124.3	118.0	129.9	250	0.277505
14	150	134.5	130.1	148.6	148.8	124.0	141.7	710	0.301956
15	180	143.4	137.6	164.0	162.6	130.0	154.6	2,256	0.327931
16	250	153.3	145.1	181.1	181.9	136.0	168.6	2,643	0.358676
17	325	164.1	152.6	199.9	209.1	142.0	184.0	1,304	0.395719
18	400	176.0	160.1	220.7	235.0	148.0	200.7	2,183	0.441571
								442,490	

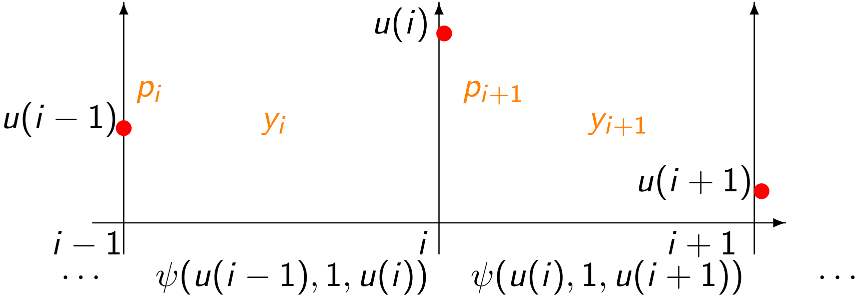
Example - Portuguese data

j \ *Year*	0	2	5	10	stationarity
1	0.39362	0.63576	0.67436	0.71953	0.73121
2	0.24659	0.05681	0.04089	0.05678	0.04913
3	0.09658	0.06896	0.08045	0.04929	0.05394
4	0.06584	0.05938	0.06980	0.04927	0.05941
5	0.05363	0.01651	0.01678	0.01904	0.01871
6	0.00958	0.05774	0.01747	0.02528	0.01678
7	0.00623	0.03644	0.03326	0.01645	0.01411
8	0.05611	0.00444	0.01265	0.01037	0.00837
9	0.02655	0.00793	0.00659	0.00840	0.00725
10	0.00037	0.02375	0.00878	0.00977	0.00612
11	0.00651	0.00480	0.01277	0.00576	0.00495
12	0.01725	0.00188	0.00596	0.00451	0.00453
13	0.00056	0.00620	0.00541	0.00407	0.00419
14	0.00160	0.00818	0.00344	0.00435	0.00398
15	0.00510	0.00294	0.00403	0.00332	0.00398
16	0.00597	0.00468	0.00208	0.00326	0.00410
17	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Expected Premium's behavior (P=115,838,792)



One run


$u(i)$
$u(i-1)$
p_i
y_i
p_{i+1}
y_{i+1}
$u(i+1)$
$i-1$
i
$i+1$
$\cdots$
$\psi(u(i-1), 1, u(i))$
$\psi(u(i), 1, u(i+1))$
$\cdots$

Probability of ruin given a initial surplus u

		$\widehat{\psi}(u,t)$ (%)							
u	t	P_0	C	N	LN	GN	B	LB	GB
350,000	2	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
	5	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
	10	**1.246**	60.502	99.999	100	96.129	25.665	38.028	50.392
1,500,000	2	0	14.807	99.954	100	75.749	**0.953**	3.407	8.374
	5	0	14.807	99.969	100	75.749	**0.953**	3.407	8.374
	10	0	14.807	99.969	100	75.749	**0.953**	3.407	8.374
2,000,000	2	0	7.569	99.854	100	63.277	0.233	**1.186**	3.716
	5	0	7.569	99.902	100	63.277	0.233	**1.186**	3.716
	10	0	7.569	99.902	100	63.277	0.233	**1.186**	3.716
2,550,000	2	0	3.495	99.601	100	48.724	0.052	0.369	**1.463**
	5	0	3.495	99.716	100	48.724	0.052	0.369	**1.463**
	10	0	3.495	99.717	100	48.724	0.052	0.369	**1.463**
3,250,000	2	0	**1.206**	98.790	100	31.551	0.009	0.083	0.426
	5	0	**1.206**	99.150	100	31.551	0.009	0.083	0.426
	10	0	**1.206**	99.154	100	31.551	0.009	0.083	0.426
6,400,000	2	0	0.008	74.639	99.923	**1.245**	0	0	0.003
	5	0	0.008	81.050	99.986	**1.245**	0	0	0.003
	10	0	0.008	81.163	99.989	**1.245**	0	0	0.003
15,000,000	2	0	0	**0.360**	31.246	0	0	0	0
	5	0	0	**1.580**	70.362	0	0	0	0
	10	0	0	**1.623**	74.868	0	0	0	0
25,000,000	2	0	0	0	**0.002**	0	0	0	0
	5	0	0	0	**0.879**	0	0	0	0
	10	0	0	0	**2.037**	0	0	0	0

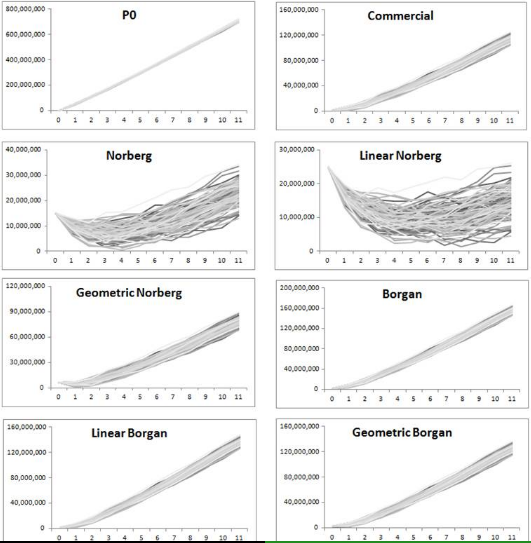
Table: $n = 2$, 5 and 10 years for each bonus scale (results in percentage).

Average of the within the year ruin probabilities

u	350,000	3,250,000	15,000,000	25,000,000	6,400,000	1,500,000	2,000,000	2,550,000
b	P_0	C	N	LN	GN	B	LB	GB
$\psi(u(i-1), 1, u(i))$								
0	**0.01245**	**0.011341**	2.028E-11	1.18E-132	0.00862	**0.0095224**	**0.0117586**	**0.0142202**
1	0	0.0054895	0.0003543	2.958E-09	**0.0109755**	0.0001391	0.0014271	0.0043680
2	0	7.113E-06	0.0035845	1.728E-05	0.0002886	3.707E-30	1.211E-14	1.447E-08
3	0	3.073E-47	0.0095027	0.0012845	9.317E-11	0	5.55E-105	9.214E-65
4	0	1.73E-170	0.0113343	0.0048442	8.457E-27	0	0	0
5	0	0	**0.0080704**	0.0080609	1.054E-68	0	0	0
6	0	0	0.0042740	0.0112567	0	0	0	0
7	0	0	0.0021026	**0.0136769**	0	0	0	0
8	0	0	0.0006633	0.0114318	0	0	0	0
9	0	0	0.0001815	0.0070086	0	0	0	0
10	0	0	5.565E-05	0.0041382	0	0	0	0

Table: By each bonus scale and for the respective initial surplus.

Some runs


P0
800,000,000
600,000,000
400,000,000
200,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11
Commercial
160,000,000
120,000,000
80,000,000
40,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11
Norberg
40,000,000
30,000,000
20,000,000
10,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11
Linear Norberg
30,000,000
20,000,000
10,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11
Geometric Norberg
120,000,000
90,000,000
60,000,000
30,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11
Borgan
200,000,000
160,000,000
120,000,000
80,000,000
40,000,000
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0 1 2 3 4 5 6 7 8 9 10 11
Linear Borgan
160,000,000
120,000,000
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0
0 1 2 3 4 5 6 7 8 9 10 11
Geometric Borgan
160,000,000
120,000,000
80,000,000
40,000,000
0
0 1 2 3 4 5 6 7 8 9 10 11

Conclusions

- Recall: The portfolio is closed for ingoing and outgoing of policyholders;
- The introduction of a BMS **increases the ruin probabilities** but, with our model, we can estimate the magnitude of that increasing;
- The average ruin probability within the year, may **foresee the time where the ruin probability is going to a intolerable risk level** and prepare a tariff revision or increase the reserve amounts.

Conclusions

- No perceptible changes in the scales can have a **big impact on initial surplus** u in order to have an acceptable ruin probability.
- The model may also be useful as a way to **estimate the amount needed for initial capital** u.
- The model provides simple and effective methodology for **assessing scales and bonus malus schemes**.
- It can be applied also for the Solvency II purposes to obtain the estimates of ruin probabilities in **one year period**.

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